And one more limit of Lalescu kind sequence.
https://www.linkedin.com/feed/update/urn:li:activity:6592451179303841792
5495: **Proposed by D. M. Bătinetu-Giurgiu**, **"Matei Basarab" National College**, **Bucharest and Neculai Stanciu**,“**George Emil Palade**” **School**, **Buzău**, **Romania**
Let $\{x_n\}_{n\geq 1}$, $x_1 = 1, x_n = 1 \cdot \sqrt{3!!} \cdot \sqrt[3]{5!!} \cdot \ldots \cdot \sqrt[n]{(2n-1)!!}$.
Find:

$$L := \lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{x_{n+1}}} - \frac{n^2}{\sqrt[n]{x_n}}\right)$$

Solution by Arkady Alt , **San Jose** ,**California**, **USA**.
First we will find two limits which we will need further, namely:

a) $\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n}$;

b) $\lim_{n\to\infty}\frac{\sqrt[n]{x_n}}{n}$

To find them we will use the special case of multiplicative version of Cesaro's Theorem (or, Geometric Mean Limit Theorem (GML Theorem)).
Theorem*:
Let $\{a_n\}_{n\geq 1}$ be sequence of positive real numbers such that $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = a$.
Then $\lim_{n\to\infty}\sqrt[n]{a_n} = a$.

a) Let $a_n := \frac{(2n-1)!!}{n^n}, n \in \mathbb{N}$. Since $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(\frac{(2n+1)!!}{(n+1)^{n+1}} \cdot \frac{n^n}{(2n-1)!!}\right) =$
$\lim_{n\to\infty}\left(\frac{2n+1}{n+1} \cdot \frac{1}{\left(1+\frac{1}{n}\right)^n}\right) = \frac{2}{e}$ then by GML Theorem

$$\lim_{n\to\infty}\frac{\sqrt[n]{(2n-1)!!}}{n} = \lim_{n\to\infty}\sqrt[n]{a_n} = \frac{2}{e};$$

b) Let $a_n := \frac{x_n}{n^n}., n \in \mathbb{N}$. Since $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(\frac{x_{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{x_n}\right) =$
$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(2n+1)!!}}{n+1} \cdot \frac{1}{\left(1+\frac{1}{n}\right)^n}\right) = \frac{2}{e} \cdot \frac{1}{e} = \frac{2}{e^2}$ then by GML Theorem

$$\lim_{n\to\infty}\frac{\sqrt[n]{x_n}}{n} = \lim_{n\to\infty}\sqrt[n]{a_n} = \frac{2}{e^2}.$$

Now we ready to find L.
Since $\lim_{n\to\infty}\frac{n}{\sqrt[n]{x_n}} = \frac{e^2}{2}$ then $L = \lim_{n\to\infty}\frac{n^2}{\sqrt[n]{x_n}}\left(\frac{(n+1)^2}{n^2} \cdot \frac{\sqrt[n]{x_n}}{\sqrt[n+1]{x_{n+1}}} - 1\right) =$
$\frac{e^2}{2}\lim_{n\to\infty} n(e^{\alpha_n} - 1)$, where $\alpha_n := \ln\left(\frac{(n+1)^2}{n^2} \cdot \frac{\sqrt[n]{x_n}}{\sqrt[n+1]{x_{n+1}}}\right), n \in \mathbb{N}$.
Noting that $\lim_{n\to\infty}\alpha_n = 0$ (because $\lim_{n\to\infty}\left(\frac{(n+1)^2}{n^2} \cdot \frac{\sqrt[n]{x_n}}{\sqrt[n+1]{x_{n+1}}}\right) =$

$\lim\limits_{n\to\infty}\dfrac{\frac{\sqrt[n]{x_n}}{n}}{\frac{\sqrt[n+1]{x_{n+1}}}{n+1}} \cdot \lim\limits_{n\to\infty}\dfrac{n+1}{n} = 1$) and, therefore, $\lim\limits_{n\to\infty}\dfrac{e^{\alpha_n}-1}{\alpha_n}$ we obtain

$L = \dfrac{e^2}{2}\lim\limits_{n\to\infty} n(e^{\alpha_n}-1) = \dfrac{e^2}{2}\lim\limits_{n\to\infty} n\alpha_n \cdot \dfrac{e^{\alpha_n}-1}{\alpha_n} = \dfrac{e^2}{2}\lim\limits_{n\to\infty} n\alpha_n =$

$\dfrac{e^2}{2}\ln\left(\lim\limits_{n\to\infty}\left(\dfrac{(n+1)^2}{n^2}\cdot\dfrac{\sqrt[n]{x_n}}{\sqrt[n+1]{x_{n+1}}}\right)^n\right).$

Since $\left(\dfrac{(n+1)^2}{n^2}\cdot\dfrac{\sqrt[n]{x_n}}{\sqrt[n+1]{x_{n+1}}}\right)^n = \dfrac{(n+1)^{2n}}{n^{2n}}\cdot\dfrac{x_n}{x_{n+1}}\cdot\sqrt[n+1]{x_{n+1}} =$

$\left(1+\dfrac{1}{n}\right)^{2n}\cdot\dfrac{\sqrt[n+1]{x_{n+1}}}{\sqrt[n+1]{(2n+1)!!}} = \left(1+\dfrac{1}{n}\right)^{2n}\cdot\dfrac{\sqrt[n+1]{x_{n+1}}}{n+1}\cdot\dfrac{n+1}{\sqrt[n+1]{(2n+1)!!}}$

then $\lim\limits_{n\to\infty}\left(\dfrac{(n+1)^2}{n^2}\cdot\dfrac{\sqrt[n]{x_n}}{\sqrt[n+1]{x_{n+1}}}\right)^n = e^2\cdot\dfrac{2}{e^2}\cdot\dfrac{e}{2} = e.$

Thus, $L = \dfrac{e^2}{2}\ln e = \dfrac{e^2}{2}.$

* **GML Theorem.(Multiplicative version of Cesaro's Theorem)**

Let $\{b_n\}_{n\geq 1}$ be sequence of positive real numbers such that $\lim\limits_{n\to\infty} b_n = b$.

Then $\lim\limits_{n\to\infty}\sqrt[n]{b_1b_2\ldots b_n} = b$.

Applying this theorem to $b_n = \dfrac{a_n}{a_{n-1}}, n\geq 2$ and $b_1 = a_1$ we obtain that

$\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n} = a$ implies $\lim\limits_{n\to\infty}\sqrt[n]{a_n} = a$.

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